ON THE STABILITY OF THE EQUILIBRIUM OF A PHYSICAL LIQUID-FILLED PENDULUM

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In paper [1] Rumiantsev proves a theorem whereby the stability of the equilibrium position of a hollow body filled with two homogeneous nonmixing liquids with surface tension property, is guaranteed if the functional

$$F = V + \alpha S + \alpha_1\sigma_1 + \alpha_2\sigma_2, \qquad V = V_0 + \iiint_{\tau_1} \rho_1 V_1 d\tau + \iiint_{\tau_2} \rho_2 V_2 d\tau$$

has an isolated minimum F_0 for the equilibrium position.

Here V is the potential energy of the external forces applied to the body (V_0) and liquids; τ_1, τ_2 are the volumes occupied by the first and second liquids, respectively; S is the interface area of the liquids; σ_1, σ_2 are the areas of contact of the liquids with the cavity wall; α, α_1, α_2 are the corresponding coefficients of surface tension; ρ_1, ρ_2 are the densities, $\rho_1 > \rho_2$.

A necessary condition for a minimum in the functional F is that it has a weak minimum. The present paper gives sufficient conditions for a weak minimum in F when the body has the form of a heavy physical pendulum with a cylindrical cavity filled with two heavy liquids. Here the problem of a weak minimum in F reduces to finding the conditions of a minimum in the function G of the body's coordinates. Such a device was employed by Pozharitskii [2] in solving a similar minimum problem for a body containing a liquid devoid of surface tension.

Let there be an equilibrium position at which the interface of the liquids lies at a finite distance from the end faces of the cavity and intersects the side walls, which will be assumed vertical for simplicity. The heavier liquid occupies the lower part of the cavity.

We introduce the stationary coordinate axes $x_1 y_1 z_1$ as follows: the axis z_1 is directed upward along the vertical, y_1 is the axis of the pendulum, and x_1 is orthogonal to the plane $z_1 y_1$. The movable axes xyz linked with the solid portion of the pendulum are: the z-axis parallel to the cavity wall, the x-axis in the plane of swing, and the y-axis coincident with y_1 .

To abbreviate our notation we introduce the notation

$$f_x = \frac{\partial f}{\partial x}, \qquad \Phi^x(f_i) = \frac{f_{i,x}}{\sqrt{1 + f_{i,x}^2 + f_{i,y}^2}}$$

$$P^x(\zeta) = \frac{(1 + f_{1,y}^2)\zeta_x - f_{1,x} f_{1,y} \zeta_y}{(1 + f_{1,x}^2 + f_{1,y}^2)^{3/2}} \qquad (i = 0, 1), \quad (xy)$$

1. Let us indicate the method of obtaining the function G. In the equilibrium position the interface $z = f_0(x, y)$ is determined from Equation

$$g(\rho_1 - \rho_2) f_0 - \alpha \{[\Phi^x(f_0)]_x + [\Phi^y(f_0)]_y\} = \text{const} = c_0 \qquad (1)$$

with the boundary condition

$$\Phi^x(f_0)\frac{dy}{dl} - \Phi^y(f_0)\frac{dx}{dl} = \frac{\alpha_2 - \alpha_1}{\alpha}$$

along the line l of intersection of the cavity walls with the plane xy.

Let the body be inclined at a small angle ϑ from the equilibrium position. The equation of the interface $z = f_1(x, y, \vartheta)$, for which $\delta F = 0$ for a fixed ϑ,

$$g(\rho_1 - \rho_2)(f_1 \cos\vartheta - x \sin\vartheta) - \alpha \{[\Phi^x(f_1)]_x + [\Phi^y(f_1)]_y\} = \text{const} = C_1 \qquad (2)$$

with the boundary condition

$$\Phi^x(f_1)\frac{dy}{dl} - \Phi^y(f_1)\frac{dx}{dl} = \frac{\alpha_2 - \alpha_1}{\alpha}$$

Let us suppose that $f_0(x, y)$ and $f_1(x, y, \vartheta)$ are single-valued functions of their coordinates with bounded derivatives, and that the differences $(f_1 - f_0)$, $(f_{1,x} - f_{0,x})$, $(f_{1,y} - f_{0,y})$ are small for small ϑ, or, more precisely, $\sim\vartheta$.

We must now prove that the functional F has a weak minimum for a fixed ϑ if the interface is given by Equation (2). To do this we can make use of the second variation of F.

The functional F clearly depends on ϑ and on the shape of the interface $z = f(x, y)$. As our measure of the inclination of the interface we take $\zeta = f - f_1$, i.e. its displacement along the z-axis. Assuming ζ and its derivatives to be small and ϑ, to be fixed, we obtain the following expression for the second variation:

$$2\delta^2 F = \iint\limits_{(\Omega)} \{g(\rho_1 - \rho_2)\cos\vartheta\,\zeta^2 + \alpha[\zeta_x P^x(\zeta) + \zeta_y P^y(\zeta)]\}\, d\Omega$$

Here (Ω) is a region of area Ω bounded by the curve l. Usually $\delta_2 F$ also includes a curvilinear integral over l. In our case this integral is lacking due to the shape of the cavity and the choice of the measure of inclination.

It is clear that

$$\delta^2 F > K \iint\limits_{(\Omega)} (\zeta^2 + \text{grad}^2\zeta)\, d\Omega \qquad \left(2K = \min_{(\Omega)} \left\{(\rho_1 - \rho_2) g \cos\vartheta,\ \frac{\alpha}{(1 + f_{1,x}^2 + f_{1,y}^2)^{3/2}}\right\}\right)$$

This is sufficient in order for F to have a minimum $G(\vartheta) = F(\vartheta, f_1)$ for a fixed ϑ, since

$$F(\vartheta, f) - G(\vartheta) = \delta^2 F + \beta \iint\limits_{(\Omega)} \text{grad}^2\zeta\, d\Omega \qquad (\beta \to 0, \quad \text{it} \quad \max_{(\Omega)}\{\zeta_x, \zeta_y\} \to 0)$$

We choose $\epsilon > 0$ such that $\beta < K/2$ is fulfilled for

$$|\max_{(\Omega)}\{\zeta_x, \zeta_y\}| < \varepsilon$$

Then

$$F(\vartheta, f) - G(\vartheta) > \tfrac{1}{2} K \int\limits_{(\Omega)} (\zeta^2 + \text{grad}^2\zeta)\, d\Omega > 0 \qquad (\zeta \not\equiv 0)$$

This enables us to prove the following theorem.

T h e o r e m . The functional F has an isolated minimum F_0 if and only if the function $G(\vartheta)$ has an isolated minimum F_0.

The proof here is similar to that given in [3].

S u f f i c i e n c y . Let $G > F_0$ for $\vartheta \neq 0$; since $F(\vartheta, f) > G(\vartheta)$ for all ϑ and $\zeta \not\equiv 0$, then certainly $F > F_0$, provided at least one of the two following conditions is fulfilled: $\vartheta \neq 0$, $\zeta \not\equiv 0$.

N e c e s s i t y . Let $F > F_0$ for all positions of the system close to the equilibrium position. Then $G > F_0$ as well, since the position of the system described by the interface $z = f_1$ and the small angle $\vartheta \neq 0$ is also close.

2. We can now derive the condition for a minimum in G by using the quadratic portion of the increment $G - F_0$. To do this, we break down the latter into two parts,

$$G - F_0 = \Delta_1 F + \Delta_2 F$$

Here $\Delta_1 F$ is the increment in F when the system as a single solid body is inclined at the angle ϑ, and $\Delta_2 F = -\delta^2 F$ is the increment in F upon displacement of the interface from the position $z = f_0$ to the position $z = f_1$, i.e. for $\zeta_1 = f_1 - f_0$.

The equation for ζ_1 is obtained by subtracting (1) from (2) and retaining only terms of the first order of smallness in ϑ and ζ_1

$$g(\rho_1 - \rho_2)(\zeta_1 - x\vartheta) - \alpha\{[P^x(\zeta_1)]_x + [P^y(\zeta_1)]_y\} = \text{const} = c_1 - c_0 \tag{3}$$

with the boundary condition

$$P^x(\zeta_1)\frac{dy}{dl} - P^y(\zeta_1)\frac{dx}{dl} = 0$$

To compute $C_1 - C_0$ we integrate (3) over the region (Ω)

$$c_1 - c_0 = -\frac{g(\rho_1 - \rho_2)}{\Omega}\iint\limits_{(\Omega)} x\vartheta\, d\Omega$$

(the integral of ζ over (Ω) is equal to zero by virtue of the incompressibility of liquids).

It is clear that in the expression for $\delta^2 F$ we can replace f_1 by f_0 by virtue of the assumptions made as regards their closeness.

Multiplying (3) by ζ_1 and integrating over the region (Ω), we obtain

$$\Delta_2 F = -\frac{1}{2}(\rho_1 - \rho_2) g \iint\limits_{(\Omega)} x\vartheta\zeta_1\, d\Omega$$

In Equation (3) we can make the substitution $\zeta_1 = \vartheta\varphi(x, y)$, . Then

$$G - F_0 = \frac{1}{2} g\vartheta^2\Big[-Mz_0 - (\rho_1 - \rho_2)\iint\limits_{(\Omega)} x\varphi\, d\Omega\Big]$$

In order for G to have a minimum, it is sufficient that

$$Mz_0 + (\rho_1 - \rho_2)\iint\limits_{(\Omega)} x\varphi\, d\Omega < 0 \tag{4}$$

Here M is the mass and z_0 the coordinate of the center of mass of the entire system in the equilibrium position.

With $\alpha = \alpha_1 = \alpha_2 = 0$, the quantity φ and condition (4) are as follows:

$$\varphi = x - \frac{1}{\Omega}\iint\limits_{(\Omega)} x\, d\Omega, \qquad Mz_0 < (\rho_2 - \rho_1)\Big[\iint\limits_{(\Omega)} x^2\, d\Omega - \Big(\iint\limits_{(\Omega)} x\, d\Omega\Big)^2\Big]$$

i.e. they coincide with the analogous condition obtained in [2].

In the particular case where the cavity is a circular cylinder of radius R, the axis of the cavity passes through the point of attachment, and $\alpha_1 = \alpha_2$, condition (4) can be written in finite form

$$Mz_0 < -\pi\Big[(\rho_1 - \rho_2)\frac{R^4}{4} - \frac{\alpha R^2}{g}\frac{I_2(\lambda R)}{I_2(\lambda R) + (\lambda R)^{-1} I_1(\lambda R)}\Big] \qquad \Big(\lambda = \frac{\sqrt{g(\rho_1 - \rho_2)}}{\sqrt{\alpha}}\Big)$$

Here $I_i(\lambda R)$ is a modified Bessel function.

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BIBLIOGRAPHY

1. Rumiantsev, V.V., Ob ustoichivosti dvizheniia tverdogo tela s zhidkost'iu, obladaiushchei poverkhnostnym natiazheniem (On the stability of motion of a rigid body containing a fluid possessing surface tension). *PMM* Vol.28, № 4, 1964.

2. Pozharitskii, G.K., Zadacha minimuma v zadache of ustoichivosti ravnovesiia tverdogo tela s chastichnym zhidkim napolneniem (The minimum problem in the problem of the stability of equilibrium of a solid body partially filled with a liquid). *PMM* Vol.26, № 4, 1962.

3. Pozharitskii, G.K. and Rumiantsev, V.V., Zadacha minimuma v voprose ob ustoichivosti dvizheniia tverdogo tela s polost'iu, zapolnennoi zhidkost'iu (The problem of the minimum in the question of the stability of motion of a solid body with a liquid-filled cavity). *PMM* Vol.27, № 1, 1963.

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